Math 251X - Exam 1 - Solutions

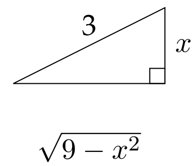
1. (6 points) Evaluate the following expressions.

a) $\arcsin(\sin(4\pi/3)))$

$= \arcsin(\frac{\sqrt{3}}{2})$

$= -\frac{\pi}{3}$

b) $\tan(\arcsin(x/3)) = \dfrac{x}{\sqrt{9-x^2}}$



2. (6 points) Solve each equation for x.

a) $e^{\sqrt{x+1}} - 2 = 5$

$e^{\sqrt{x+1}} = 7$

$\sqrt{x+1} = \ln 7$

$x + 1 = \ln^2 7$

$x = \ln^2 7 - 1$

b) $\ln x + \ln(x-3) = 0$

$\ln(x^2 - 3x) = 0$

$x^2 - 3x = 1$

$x^2 - 3x - 1 = 0$

$x = \dfrac{3 \pm \sqrt{13}}{2}$

$x = \dfrac{3 + \sqrt{13}}{2}$

3.(4 points) A contractor purchases a piece of equipment for \$18,000 that costs an average of \$ 7.50 per hour for fuel and maintenance. The equipment operator is paid \$16.50 per hour, and customers are charged \$30 per hour. a) Write an equation for the cost C of operating this equipment for t hours.

$C(t) = 18000 + 7.50t + 16.50t$

$C(t) = 18000 + 24t$

b) Write an equation for the revenue R derived from t hours of use.

$R(t) = 30t$

c) Find the break even point by finding the time at which $R = C$.

$30t = 18000 + 24t$

$6t = 18000$

$t = 3000$ hours

4. (3 points) Expand and simplify the expression $\ln\left(\dfrac{e^8\sqrt{x^2-9}}{(x^7+4)^3}\right)$

$\ln(e^8\sqrt{x^2-9}) - \ln((x^7+4)^3)$

$\ln e^8 + \ln\sqrt{x^2-9} - \ln((x^7+4)^3)$

$8 + \frac{1}{2}\ln(x+3)(x-3) - 3\ln(x^7+4)$

$8 + \frac{1}{2}\ln(x+3) + \frac{1}{2}\ln(x-3) - 3\ln(x^7+4)$

5. (6 points) Given $f(x) = \sin x$ and $g(x) = \ln x$ find the following functions and their domains.

a) $f \circ g$

$(f \circ g)(x) = \sin(\ln x)$

Domain:$(0, \infty)$

b) $g \circ f$

$(g \circ f)(x) = \ln(\sin x)$

Domain: $(2n\pi, (2n+1)\pi)$

6. (5 points) Find the inverse of $f(x) = \dfrac{3x-2}{2x+5}$ and state the domain and range of f^{-1}.

$$x = \frac{3y-2}{2y+5}$$
$$2xy + 5x = 3y - 2$$
$$2xy - 3y = -5x - 2$$
$$y(2x-3) = -5x - 2$$
$$y = \frac{-5x-2}{2x-3}$$

Since f is one-to-one:

$$f^{-1} = \frac{-5x-2}{2x-3}$$
$$D_{f^{-1}} : (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$$
$$R_{f^{-1}} : (-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, \infty)$$

7. (4 points) Evaluate the following limits. If the limit does not exist, explain why.

a) $\lim\limits_{x \to 2} 2^{\sqrt{x^2+3x+6}}$

$= 2^{\sqrt{(2)^2+3(2)+6}}$

$= 2^{\sqrt{16}}$

$= 2^4$

$= 16$

b) $\lim\limits_{x \to 3} \sec\left(\dfrac{\pi x}{2}\right)$

Notice $\sec\left(\dfrac{3\pi}{2}\right)$ is undefined

So consider the left and right hand limits

$$\lim_{x \to 3^+} \sec\left(\frac{\pi x}{2}\right) = +\infty$$
$$\lim_{x \to 3^-} \sec\left(\frac{\pi x}{2}\right) = -\infty$$

So, $\lim\limits_{x \to 3} \sec\left(\dfrac{\pi x}{2}\right)$ DNE

8. (10 points) Evaluate the following limits.

a) $\lim\limits_{x \to -2} \dfrac{x^2 - 3x - 10}{3x^2 + 5x - 2}$

$$= \lim_{x \to -2} \frac{(x-5)(x+2)}{(3x-1)(x+2)}$$
$$= \lim_{x \to -2} \frac{x-5}{3x-1}$$
$$= \frac{-2-5}{3(-2)-1}$$
$$= \frac{-7}{-7}$$
$$= 1$$

b) $\lim\limits_{x \to 3} \dfrac{x - \sqrt{x+6}}{x^3 - 3x^2}$

$$= \lim_{x \to 3} \frac{x - \sqrt{x+6}}{x^3 - 3x^2} \cdot \frac{x + \sqrt{x+6}}{x + \sqrt{x+6}}$$
$$= \lim_{x \to 3} \frac{x^2 - x + 6}{x^2(x-3)(x+\sqrt{x+6})}$$
$$= \lim_{x \to 3} \frac{(x-3)(x+2)}{x^2(x-3)(x+\sqrt{x+6})}$$
$$= \lim_{x \to 3} \frac{(x+2)}{x^2(x+\sqrt{x+6})}$$
$$= \frac{3+2}{3^2(3+\sqrt{3+6})}$$
$$= \frac{5}{54}$$

9. (10 points) Evaluate the following limits.

a) $\lim_{x \to 0} \frac{1 - e^{-x}}{e^x - 1}$

$= \lim_{x \to 0} \frac{e^x - 1}{e^x(e^x - 1)}$

$= \lim_{x \to 0} \frac{1}{e^x}$

$= \frac{1}{e^0}$

$= 1$

b) $\lim_{x \to \pi/2} \frac{\cos^2 x}{\sin x - 1}$

$= \lim_{x \to \pi/2} \frac{1 - \sin^2 x}{\sin x - 1}$

$= \lim_{x \to \pi/2} \frac{(1 - \sin x)(1 + \sin x)}{\sin x - 1}$

$= \lim_{x \to \pi/2} (-1 - \sin x)$

$= -1 - 1$

$= -2$

10. (8 points) Evaluate the following limits.

a) $\lim_{x \to -\infty} \frac{\sqrt{x^2 - 9}}{2x + 5}$

$= \lim_{x \to -\infty} \frac{\sqrt{1 - \frac{9}{x^2}}}{-2 - \frac{5}{x}}$

$= \frac{\sqrt{1 - 0}}{2 + 0}$

$= -\frac{1}{2}$

b) $\lim_{x \to \infty} (\sqrt{x^2 + 4x + 1} - x)$

$= \lim_{x \to \infty} (\sqrt{x^2 + 4x + 1} - x) \cdot \frac{(\sqrt{x^2 + 4x + 1} + x)}{(\sqrt{x^2 + 4x + 1} + x)}$

$= \lim_{x \to \infty} \frac{x^2 + 4x + 1 - x^2}{(\sqrt{x^2 + 4x + 1} + x)}$

$= \lim_{x \to \infty} \frac{4x + 1}{(\sqrt{x^2 + 4x + 1} + x)}$

$= \lim_{x \to \infty} \frac{4 + \frac{1}{x}}{(\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + 1}$

$= \frac{4}{(\sqrt{1} + 1}$

$= 2$

11. (6 points) Find the following infinite limits. Explain your reasoning.

a) $\lim_{x \to 4^+} \frac{5 - x}{x - 4}$

Since the numerator goes to 1,
and the denominator goes to 0

Then $\lim_{x \to 4^+} \frac{5 - x}{x - 4} = \infty$

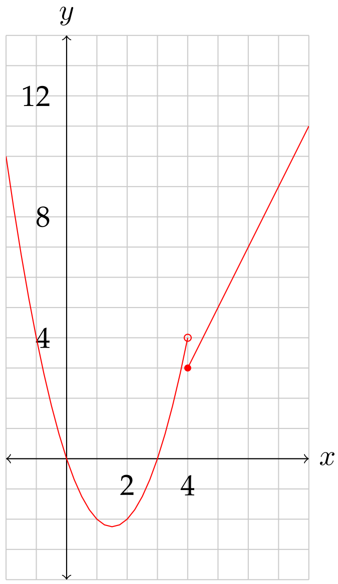
b) $\lim_{x \to 9} \frac{\sqrt{x}}{(x - 9)^4}$

Since the numerator goes to 3,
and the denominator goes to 0,

Then $\lim_{x \to 9} \frac{\sqrt{x}}{(x - 9)^4} = \infty$

12. (4 points) Given the function $f(x) = \begin{cases} x^2 - 3x & \text{if} \quad x < 4 \\ 2x - 5 & \text{if} \quad x \geq 4 \end{cases}$

a) Sketch the graph of $f(x)$



(8 points) Also, evaluate the following.

b) $f(4) = 3$

c) $f(0) = 0$

d) $f(-1) = 4$

e) $\lim_{x \to -4} f(x) = 28$

f) $\lim_{x \to 0} f(x) = 0$

g) $\lim_{x \to 4} f(x) = DNE$

h) $\lim_{x \to -\infty} f(x) = \infty$

i) $\lim_{x \to \infty} f(x) = \infty$

13. (6 points) Find any discontinuities of the following functions. Then, using the definition of continuity explain why the function is discontinuous at each point of discontinuity.

a) $f(x) = \dfrac{3x^2 - x - 2}{x - 1}$

$f(x) = \dfrac{(3x + 2)(x - 1)}{x - 1}$

This has a removable discontinuity at $x = 1$ because the function is not defined at $x = 1$.

b) $g(x) = \begin{cases} \sqrt{-x} & \text{if} \quad x < 0 \\ 3 - x & \text{if} \quad 0 \leq x \leq 3 \\ (x - 3)^2 & \text{if} \quad x > 3 \end{cases}$

This function has a non-removable discontinuity at $x = 0$ because $\lim_{x \to 0^-} g(x) = 0$ but $\lim_{x \to 0^+} g(x) = 3$. So $\lim_{x \to 0} g(x) = DNE$.

14. (4 points) For what value of the constant b is the function $f(x) = \begin{cases} x^2 + bx & x \leq 5 \\ 5\sin(\frac{\pi}{2}x) & x > 5 \end{cases}$ continuous?

Since both pieces of the function are continuous on their respective domains we need the limit to exist. So,

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x)$

$\lim_{x \to 5} (x^2 + bx) = \lim_{x \to 5} 5\sin(\dfrac{\pi x}{2})$

$5^2 + 5b = 5\sin(\frac{5\pi}{2})$

$25 + 5b = 5$

$5b = -20$

$b = -4$ will make $f(x)$ continuous.

15. (10 points) Consider the function $f(x) = x^2 + 4x - 1$.

a) (2 points) Find the average rate of change of $f(x)$ on the interval from $[1, 3]$.

$$
\begin{aligned}
f_{avg} &= \frac{f(3) - f(1)}{3 - 1} \\
&= \frac{(3^2 + 4(3) - 1) - (1^2 + 4(1) - 1)}{2} \\
&= \frac{(9 + 12 - 1) - (1 + 4 - 1)}{2} \\
&= \frac{20 - 4}{2} \\
&= \frac{16}{2} \\
f_{avg} &= 8
\end{aligned}
$$

b) (5 points) Use the definition of the derivative to find $f'(x)$. **You must use the definition or no credit will be given.**

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h \to 0} \frac{[(x+h)^2 + 4(x+h) - 1] - [x^2 + 4x - 1]}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 4x + 4h - 1 - x^2 - 4x + 1}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 4h}{h} \\
&= \lim_{h \to 0} (2x + h + 4) \\
f'(x) &= 2x + 4
\end{aligned}
$$

c) (3 points) Find an equation of the tangent line to $f(x)$ at the point $(4, 31)$. Give your answer into slope-intercept form.

$f'(4) = 2(4) + 4 = 12$

So, $y - 31 = 12(x - 4)$

$y = 12x - 17$